Chance constrained zero-sum games

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Mots-clés : *Stochastic programming, Two-person zero-sum game, Joint probabilistic constraints, Random payoff.*

1 Introduction

Game theory has been widely studied in several fields, e.g., economics, networks, political science and psychology and computer science. Since von Neumann and Morgenstern [6] developed game theory, this topic has attracted the attention of economists, mathematicians and operations researchers. We refer the reader to some famous game theory textbooks[1]. In this paper, we study a two-person zero-sum game where the payoff matrix entries are random and the constraints are satisfied jointly with a given probability.

2 Problem formulation

We consider the classic two-person zero-sum game. There are only two players where one player wins what the other player loses. We refer to the players as Player I and Player II. Let $A = (a_{i,j})_{n\times m}$ be the payoff matrix of a two-person zero-sum game. If Player I plays pure strategy i and Player II uses pure strategy j, then the payoff from Player II to Player I is a_{ij}. In the game, Player I seeks a mixed strategy to maximize his minimum payoff. Player II seeks a mixed strategy to minimize his maximum loss. In this case, the two-person zero-sum game can be mathematically formulated as two linear programming (LP) problems :
For Player I,

$$\begin{aligned} d^* := \max \quad & d \\ s.t. \quad & \min_y (x^T A y : y^T e_m = 1, y \geq 0) \geq d \\ & x^T e_n = 1, x \geq 0 \end{aligned}$$

or

$$(P1) \quad \begin{aligned} d^* := \max \quad & d \\ s.t. \quad & A^T x \geq d e_m \\ & x^T e_n = 1, x \geq 0 \end{aligned}$$

where e_k is a $k-$dimensional vector with all elements are equal to 1.
For Player II,

$$\begin{aligned} t^* := \min \quad & t \\ s.t. \quad & \max_x (x^T A y : x^T e_n = 1, x \geq 0) \leq t \\ & y^T e_m = 1, x \geq 0 \end{aligned}$$

or

$$(P2) \quad t^* := \min \quad t$$
$$s.t. \quad Ay \leq te_n$$
$$y^T e_m = 1, y \geq 0$$

When the payoff matrix A is deterministic, the famous von Neumann minimax thoerem [5] states that the two optimal values of the two LP problems are equal, i.e., $d^* = t^*$. However, due to real-world uncertainties, the payoff matrix is unknown in advance. In this case, it is natural to model the payoff matrix by continuously or discretely distributed random variables leading to a stochastic optimization problem.

In this paper, we consider chance-constrained criteria for determining optimal strategies [2]. Each player optimizes his strategy and return such that the probability of attaining that return is at least some given value. Therefore, the random-payoff two-person zero-sum game can be formulated as the following stochastic programming problems [3, 4] :
For Player I,

$$(P3) : \quad P^* := \max_{x,d} \quad d$$
$$s.t. \quad \boldsymbol{Pr}\{A^T x \geq de_m\} \geq \alpha$$
$$x^T e_n = 1, x \geq 0$$

For Player II,

$$(P4) : \quad D^* := \min_{y,t} \quad t$$
$$s.t. \quad \boldsymbol{Pr}\{Ay \leq te_n\} \geq \beta$$
$$y^T e_m = 1, y \geq 0$$

3 Our results

We prove that for the general random-payoff zero-sum game there exists a "weak duality" between the two formulations, i.e., the optimal value of the minimizing player is an upper bound of the one of the maximizing player. Under certain assumptions, we show that there also exists a "strong duality" where their optimal values are equal. Moreover, we develop two approximation methods to solve the game problem when the payoff matrix entries are independent and normally or elliptically distributed.

Références

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