Characterization of optimal solutions for the $1||L_{\max}$ problem

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1 Introduction

Many scheduling problems can be solved in polynomial time by applying a simple rule, often called "priority rule" [Smith, 1956]. Application of these rules indicates whether it is preferable to place a job J_i before a job J_j. For example, the problem $1||\sum C_j$ is solved optimally by sorting the jobs according to their non decreasing processing time order [6], the problem $1||L_{max}$ is solved optimally by sorting the jobs according to their non decreasing due dates order [4], the problem $F2||C_{max}$ is solved by the well-known Johnson's rule [5], etc.

It is also well known that each scheduling problem generally has so many optimal solutions (a potentially exponential number). Some preliminary studies concerning the search of the characteristics of these optimal solutions (characteristics but not the list) have been conducted in this direction, using the lattice of permutations as support [1, 2, 3]. In these papers, some open problems are identified : (1) finding a sequence of jobs as deep as possible in the lattice (i.e. with minimum level), (2) finding a sequence of jobs as deep as possible in the lattice that has not been characterized yet.

In this paper, we consider the $1||L_{\max}$ problem, and we focus on problem (1). We show that the expression of the level presents similarities with a new type of objective function, involving the (weighted) position of a job in the sequence (and not its completion time). We show that the minimization of a general function of weighted positions is strongly NP-hard and give an exponential dynamic programming algorithm. We propose also a backward heuristic algorithm.

2 Position of the problem

We consider a set of n jobs J_j, $1 \leq j \leq n$ with processing times p_j and due dates d_j. We minimize the maximum lateness $L_{\max}$, by sorting the jobs according to EDD rule. With a preliminary treatment in $O(n \log n)$, we renumber the jobs in EDD order, and we modify the due dates, defining deadlines $\tilde{d}_j = d_j + L^*_{\max}$, $\forall j \in \{1, ..., n\}$. Therefore, finding a sequence with optimal maximum lateness and minimum level in the lattice is now equivalent to find a feasible sequence with minimum level in the lattice, respecting the deadlines.

In the lattice of permutations, the level of a sequence is equal to the number of precedences of type $J_i \prec J_j$ with $i < j$.

Let consider a sequence σ. We denote by N_j the number of jobs after J_j in σ with an index greater than j. The level of σ is given by $\sum_{j=1}^{n} N_j$ and the problem to solve can be denoted by $1|\tilde{d}_j|\sum N_j$.

Let us denote by $x_{j,k}$ a binary variable equal to 1 if J_j is in position k and 0 otherwise. We can show that $\sum_{j=1}^{n} N_j = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{i=j+1}^{n} \sum_{h=k+1}^{n} x_{i,h} x_{j,k}$. This expression is quadratic. We can show that the minimization of $Z' = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{i=j+1}^{n} \sum_{h=k+1}^{n} x_{i,h}$ (notice that it is not really the expression of $\sum_{j=1}^{n} N_j$) is equivalent to the minimization of $\sum_{j=1}^{n} \sum_{k=1}^{n} jkx_{j,k}$. In this expression, $\sum_{k=1}^{n} kx_{j,k}$ is exactly the position of J_j in σ. So if we define by P_j the position of job J_j, we have $Z' = \sum_{j=1}^{n} jP_j$. This new type of objective function never appeared in the scheduling literature before to the best of our knowledge.

3 New problems related to $\sum P_j$

3.1 Some easy problems

The new objective function related to the jobs positions is very particular. Some problems are very trivial :

- solving the $1|\tilde{d}_j|P_{\max}$ problem has no interest since $P_{\max} = n$ whatever the sequence is. Sequence $(J_1, J_2, ..., J_n)$ is optimal.
- solving the $1|\tilde{d}_j|\sum P_j$ problem has no interest since $\sum P_j$ is always equal to $n(n+1)/2$. Sequence $(J_1, J_2, ..., J_n)$ is optimal.
- solving the $1||\sum w_j P_j$ problem is trivial. This problem is equivalent to the $1|p_j = 1|\sum w_j C_j$ and it can be solved in $O(n \log n)$ time by sorting the jobs in non increasing w_j order.

The most interesting problem is the $1|\tilde{d}_j|\sum w_j P_j$ problem.

3.2 Problem $1|\tilde{d}_j|\sum w_j P_j$

Proposition : Problem $1|\tilde{d}_j|\sum w_j P_j$ is strongly NP-hard.

Proof. by reduction from 3-PARTITION problem.

An exponential time Dynamic Programming algorithm will be given as well as a heuristic procedure with performance guaranty.

4 Conclusions and Perspectives

We identify a new problem of minimization, that is not solved up to now. However, the study of this new problem has led to the definition of a new objective function for scheduling problems : the (weighted) position of jobs. The most important is that we show that the minimization of this function subject to jobs deadlines is strongly NP-hard. Some problems remain open, as the $1|\tilde{d}_j|\sum w_j P_j$ problem with $w_j = j\ \forall j$, or the $1|\tilde{d}_j|\sum N_j$ problem.

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