Stochastic models for the Kidney Exchange Program

Margarida Carvalho[1], Miguel Constantino[2], Pierre Hosteins[1], Xenia Klimentova[1]
João Pedro Pedroso[1,3], Ana Viana[1,4]

[1] INESC TEC, Rua Dr. Roberto Frias 378, 4200 Porto, Portugal
{margarida.carvalho,pierre.hosteins,xenia.klimentova,ana.viana}@inesctec.pt,jpp@fc.up.pt

[2] Departamento de Estatística e Investigação Operacional, Faculdade de Ciências da Universidade de Lisboa
mfconstantino@fc.ul.pt

[3] ISEP - School of Engineering, Polytechnic of Porto, 4200-072 Porto, Portugal

[4] Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal

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1 Introduction

Many countries have implemented kidney exchange programmes for transplanting patients with kidney failure (see e.g. [1]). These programmes allow a patient with a willing donor, physiologically incompatible with the patient, to participate in an exchange with other patient-donor pairs. A patient can then benefit from the kidney of a donor in the pool if they are found to be compatible. The problem can be represented by a directed graph $G = (V, A)$ where a node represents a patient-donor pair and a directed edge from node v_1 to v_2 represents a compatibility between the donor of pair v_1 and the patient of pair v_2. The optimisation problem consists in finding an optimal match between patients and donors with respect to a numerical criterion such as, e.g., the total number of transplants or the total number of transplants with identical blood type. The solution to the problem consists in node sets forming disjoint cycles, such that when a donor is operated, his patient receives a kidney too.

A concrete problem that arises after providing an optimal matching is due to the possibility of either a node failure (a patient or its donor pulls back from the program) or an edge failure. This last possibility comes from the fact that edges are built based on preliminary compatibility tests. Once a matching is approved, more extensive tests are performed between the selected donors and patient which sometimes reveal incompatibility. A probability of failure can thus be attributed to any node and edge in the graph. The problem then becomes stochastic and we must consider the expected number of transplants of a given solution. Different works exist to tackle a stochastic version of the Kidney Exchange Problem (KEP). For example [2] takes into account the probabilities of failure to compute the maximum expected number of transplants with different possibilites of recourse scenarios in case of failures. This approach, however, does not take into account any measure of robustness for the selected solution. Solution robustness is handled in [4], a drawback being that in such cases one only considers the worst scenario, an extremely conservative assumption. A good measure of the potential loss in the worst cases is known under the name of Conditional Value at Risk (CVaR), which represents the average value of the potential loss over a fraction of the worst scenarios [3]. Besides taking into account the potential loss of a solution, CVaR has good mathematical properties which are valuable for solving mathematical stochastic optimisation models.

2 Stochastic models for KEP including CVaR

A classic formulation of KEP relies on identifying the set $\mathcal{C}$ of potential cycles (of maximum size k) inside the graph. A binary variable x_c is introduced for each cycle $c \in \mathcal{C}$ to decide if a cycle should be selected. To model stochasticity, we add to the base model a set of scenarios $\mathcal{S}$, where each $s \in \mathcal{S}$ is defined by a certain set of nodes and arcs failing (and thus potential cycles being cancelled). A weight for each cycle w_c^s is equal to the number of transplants for cycle c in scenario s. We also consider a recourse policy, not investigated in [2], that keeps the cycles that did not fail untouched while reassigning the nodes of failed cycles to alternative cycles. The stochastic model with recourse can be modelled as :

$$\max E[Q(x,w)] \tag{1}$$

$$\sum_{c \in \mathcal{C}:\, i \in V(c)} x_c \leq 1, \qquad i \in V, \tag{2}$$

$$Q(x,w^s) = \max \sum_{c \in \mathcal{C}} w_c^s y_c^s, \tag{3}$$

$$\sum_{c \in \mathcal{C}:\, i \in V(c)} y_c^s \leq 1, \qquad s \in \mathcal{S}, i \in V, \tag{4}$$

$$y_c^s \geq x_c, \qquad s \in \mathcal{S}, c \in \mathcal{C} :\ w_c^s > 0, \tag{5}$$

with y_c^s as recourse variables, $Q(x,w)$ as the recourse function and $V(c)$ as the set of nodes in cycle $c \in \mathcal{C}$. Even though the number of scenarios is exponential in the size of the graph, since $|\mathcal{S}| = 2^{|V|+|A|}$, there are ways to decompose the graph into subcomponents in the recourse subproblem [2].

In order to introduce a certain measure of robustness in the model, we introduce CVaR in different ways, e.g. as a constraint or as a second objective. A useful mathematical formulation for CVaR is given by [3] :

$$\mathrm{CVaR}_\alpha(x) = \min_{\zeta} \left[F_\alpha(x,\zeta)\right], \tag{6}$$

$$F_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha} E\left[\max(0, L(x,w) - \zeta)\right], \tag{7}$$

$$= \zeta + \frac{1}{1-\alpha} \sum_{s \in \mathcal{S}} p_s(\max(0, L(x,w^s) - \zeta). \tag{8}$$

The function $L(x,w^s)$ represents the loss of solution x in scenario s compared to the case without failure and $1-\alpha$ represents the fraction of the worst scenarios over which we compute CVaR. Since F_α is a sum of non-linear functions, the model is usually linearised by introducing a continous positive variable η_s for each scenario. The loss function is simply the difference between the best possible outcome for solution x and the recourse result : $L(x,w^s) = \sum_{c \in \mathcal{C}} w_c^s x_c - Q(x,w^s)$. The resulting model can in certain cases be linearised to obtain a MIP that we can try to solve either exactly through the use of a linear solver or by heuristic approaches.

We consider different types of recourse policies and sampling methods over the scenarios in order to solve KEP and compare the solutions for the different recourse policies, with exact or heuristic methods.

Références

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