On Clique Interdiction Problems in Graphs

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1 Problem Definition and Notation

Given an undirected graph $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges, a *clique* is a subset $V_C \subseteq V$ of nodes that induces a complete subgraph. The *maximum clique problem* in G is to find a clique of maximum size.

Given a maximum number of nodes k which can be removed from the graph G, the *node interdiction clique problem* (NICP) is to find a subset of k nodes to delete from G so that the maximum size of the clique in the remaining graph is minimized. The problem can be seen as a Stackelberg game in which a *leader* decides on the subset of nodes to interdict (within the given interdiction budget), and after that a *follower* solves the maximum clique problem in the remaining graph. The goal of the leader is to choose an interdiction policy that will guarantee the worst possible outcome for the follower.

Let us introduce the following binary decision variables:

$$w_u = \begin{cases} 1, & \text{if node } u \text{ is interdicted by the leader ,} \\ 0, & \text{otherwise} \end{cases} \qquad \forall u \in V$$

$$x_u = \begin{cases} 1, & \text{if node } u \text{ is used in the maximum clique of the follower,} \\ 0, & \text{otherwise} \end{cases} \qquad \forall u \in V$$

Let $\mathcal{W}$ be the set of all feasible interdiction policies, i.e.:

$$\mathcal{W} = \left\{ w = (w_1, w_2, \dots, w_n) : \sum_{u \in V} w_u \le k,\ w_u \in \{0,1\}, \quad \forall u \in V \right\}. \tag{1}$$

and let $\mathcal{K}$ be the set of all feasible cliques, i.e.,

$$\mathcal{K} = \{ x = (x_1, x_2, \dots, x_n) : x_u + x_v \le 1, \forall (u,v) \notin E,\ x_u \in \{0,1\}, \forall u \in V \}. \tag{2}$$

Then, the NICP can be formulated as follows:

$$\text{(NICP)} \quad \min \max_{x \in \mathcal{K}} \sum_{u \in V} x_u \tag{3}$$

$$\text{s.t.} \quad x_u \le 1 - w_u \quad u \in V \tag{4}$$

$$\sum_{u \in V} w_u \le k \tag{5}$$

$$w_u \in \{0,1\} \quad u \in V. \tag{6}$$

A problem closely related to NICP is the Maximum Vertex Block Cliquer Problem, in which one searches for a subset of nodes of minimum cardinality in a *weighted* graph G to be removed, so that the maximum weighted clique in the remaining graph is bounded above by a given integer $r \geq 1$. This problem has been studied in [4] where an exact algorithm is proposed. The NICP also belongs to a larger family of Interdiction Problems under Monotonicity, which has been studied in [2]. The NICP problems are particularly important in the social network analysis (see, e.g., [1, 3]).

In this work we propose exact approaches for solving the NICP: we develop branch-and-cut procedures, enhanced by some preprocessing and heuristic ideas.

References

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