Polyhedral Results on the Double TSP with Multiple Stacks

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1 Introduction

In the *double TSP with multiple stacks* (*DTSPMS*), introduced in [4], n items have to be picked up in one city, stored in a vehicle having s identical stacks, and delivered to n customers in another city. Both cities are modeled by the complete digraph $D = (V, A)$ with $V = \{0, \dots, n\}$ and $A = \{(i, j) \colon i \neq j \in V\}$, where 0 represents a depot. The pickup (resp. delivery) city then corresponds to a cost vector $c^1 \in \mathbb{R}^{|A|}$ (resp. $c^2 \in \mathbb{R}^{|A|}$) on the arcs of D. Item i has to be picked up from vertex i of the first city, and delivered to vertex i of the second city. The pickup phase has to be completed before the delivery phase starts. Each phase consists in a Hamiltonian circuit performed by the vehicle which starts from the depot and visits the remaining n vertices of the graph exactly once before coming back to the depot. When a new item is picked up, it is stored on the top of a stack of the vehicle and no rearrangement of the stacks is allowed. During the delivery circuit the stacks are unloaded by following a last-in-first-out policy: only the items currently on the top of their stack can be delivered. The goal is to find a pair of s-consistent Hamiltonian circuits C_1 (for the pickup) and C_2 (for the delivery) whose cost $c^1(C_1) + c^2(C_2)$ is minimum — a pair of Hamiltonian circuits being *s-consistent* if a vehicle with s stacks can perform both while satisfying the last-in-first-out policy. We introduce an integer linear programming formulation for this problem and link its integer hull to a specific ATSP polytope as well as to a specific set covering polytope.

2 Formulation of the DTSPMS

Every Hamiltonian circuit H of D corresponds to a solution (x, y) of the following formulation, see [3] and the references therein:

$$\sum_{j \in V \setminus \{i\}} x_{ij} = 1 \qquad \text{for all } i \in V, \tag{1}$$

$$\sum_{i \in V \setminus \{j\}} x_{ij} = 1 \qquad \text{for all } j \in V, \tag{2}$$

$$y_{ij} + y_{ji} = 1 \qquad \text{for all distinct } i, j \in V \setminus \{0\}, \tag{3}$$

$$y_{ij} + y_{jk} + y_{ki} \geq 1 \qquad \text{for all distinct } i, j, k \in V \setminus \{0\}, \tag{4}$$

$$x_{ij} \leq y_{ij} \qquad \text{for all distinct } i, j \in V \setminus \{0\}, \tag{5}$$

$$y_{ij} \in \{0, 1\} \qquad \text{for all distinct } i, j \in V \setminus \{0\}, \tag{6}$$

$$x_{ij} \in \{0, 1\} \qquad \text{for all distinct } i, j \in V. \tag{7}$$

Above, $x_{ij} = 1$ whenever arc (i, j) is in H and $y_{ij} = 1$ whenever i precedes j in H (assuming that 0 is the first vertex). By using arc and precedence variables x^1, y^1 (resp. x^2, y^2) with the same meaning as above for the pickup (resp. delivery) network, it can be shown that the solutions to the DTSPMS are described by the following formulation, see *e.g.* [1]:

$$(x^t, y^t) \text{ satisfies (1)–(7)} \qquad \text{for } t = 1, 2, \tag{8}$$

$$\sum_{i=1}^{s} (y^1_{v_i v_{i+1}} + y^2_{v_i v_{i+1}}) \geq 1 \qquad \text{for all distinct } v_1, \ldots, v_{s+1} \in V \setminus \{0\}. \tag{9}$$

Polyhedral Results

Throughout, $DTSPMS_{n,s}$ denotes the convex hull of the solutions to (8)–(9) and $ATSP_n$ denotes the convex hull of the solutions to (1)–(7). The valid inequalities for $ATSP_n$ presented in [3] are also valid for $DTSPMS_{n,s}$, see [1]. Moreover, every facet of $ATSP_n$ gives two facets of $DTSPMS_{n,s}$, as expressed in the following theorem.

Theorem 1 ([1]) *For $n \geq 5$ and $s \geq 2$, if $ax + by \geq c$ defines a facet of $ATSP_n$, then $ax^t + by^t \geq c$ defines a facet of $DTSPMS_{n,s}$, for $t = 1, 2$.*

Theorem 1 characterizes a super-polynomial number of "routing" facets of $DTSPMS_{n,s}$, see the discussion in [1]. When focusing on the "consistency" requirement of the problem, we consider the polytope $SC_{n,s} = conv\{(y^1, y^2) \in \{0,1\}^{n(n-1)} \times \{0,1\}^{n(n-1)} \colon \text{(9) are satisfied}\}$. The latter is a *set covering polytope*, that is a polytope of the form $conv\{x \in \{0,1\}^d \colon Ax \geq \mathbf{1}\}$, with A being a 0,1-matrix. Using this property one can show the following result (where inequalities that consist in 0,1 bounds on the variables are called *trivial*).

Proposition 1 ([2]) *Every non-trivial facet-defining inequality of $SC_{n,s}$ is of the form $ay^1 + ay^2 \geq b$, where $ay \geq b$ is a non-trivial facet-defining inequality of the polytope $RSC_{n,s} = conv\{y \in \{0,1\}^{n(n-1)} \colon \sum_{i=1}^{s} y_{v_i v_{i+1}} \geq 1$ for all distinct $v_1, \ldots, v_{s+1} \in V \setminus \{0\}\}$.*

In the case of the double TSP with two stacks, $RSC_{n,2}$ can be expressed is the vertex cover polytope of the graph $G_n = (U, E)$ whose vertices are u_{ij} for all distinct $i, j \in V \setminus \{0\}$ and the edges are $\{u_{ij}, u_{jk}\}$ for all distinct $i, j, k \in V \setminus \{0\}$, see [2]. By defining an *odd hole* of G_n as a vertex subset inducing a chordless cycle as a subgraph, we hence obtain the following corollary.

Corollary 1 *Inequalities $y^1(H) + y^2(H) \geq \frac{|H|+1}{2}$ for all odd holes H of G_n, are valid for $DTSPMS_{n,2}$.*

Conclusions

We derived links between a polytope describing the double TSP with multiple stacks and a specific ATSP polytope as well as a specific set covering polytope. Our results are possibly exploitable in an efficient computational framework, see [1, 2].

References

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