A Heuristic for Solving the Maximum Ratio Clique Problem

Mahdi Moeini[1], Dominik Goeke[1], David Poganiuch[1]

[1] BISOR, Technical University of Kaiserslautern, Postfach 3049, D-67653 Kaiserslautern, Germany.
{mahdi.moeini,dominik.goeke,poganiuch}@wiwi.uni-kl.de

Mots-clés : *graphs, clique, heuristics, variable neighborhood search.*

1 Introduction

In this study, we are interested in a variant of clique problems that is called the *maximum ratio clique problem (MRCP)*. Suppose that each node of a given graph has two non-negative weights. For any clique, by using its node weights, we can compute a number that is the ratio of the sums of node weights. The MRCP looks for a maximal clique that has the maximum ratio. It has been proven that MRCP is NP-hard. Consequently, exact methods cannot solve, in reasonable time, the large instances of the MRCP. Our contribution consists in proposing a heuristic, based on Variable Neighborhood Search (VNS) algorithm to solve the MRCP. We evaluate the efficiency of the algorithm by using standard instances and by comparing its results with best solutions of classical methods. According to our observations, our algorithm provides high-quality solutions in short computation time and has a better performance than the classical methods.

2 Problem Description and Mathematical Modeling

Suppose that a graph $G = (V, E)$ is given, where $V = \{1, 2, \dots, n\}$ and E are the sets of nodes and edges, respectively. Any complete subgraph of G is called a clique. Assume that G_c is a clique in G, then G_c is *maximal* if it is part of none of the other cliques in G. Another class of cliques are the *maximum* cliques, i.e., the cliques (in a graph) with maximum cardinality. If any node $i \in V$ of G has the weight w_i, the *maximum weight clique problem (MWCP)* consists in finding a clique with maximum sum of node weights. We can generalize the MWCP by considering a larger set of node weights. More precisely, we consider two sets of non-negative node weights : a_i and b_i for each node $i \in V$. Then, we look for a maximal clique, like $C \subset G$, such that the quantity $\frac{\sum_{i\in C} a_i}{\sum_{i\in C} b_i}$ is maximized. This problem is called the *maximum ratio clique problem (MRCP)*. The MRCP problem can be useful in several applications, e.g., in social networks and in portfolio optimization (for more details, see [4, 5], and references therein). The MRCP is formulated as a binary fractional problem and it has been proven that it is NP-hard [5]. There is rich class of methods for solving binary fractional problems. We use some of these methods and, furthermore, propose a new approach for solving the MRCP.
In order to present the mathematical model of the MRCP, we assume that $A = (a_{ij})$ is the adjacency matrix of $G = (V, E)$ and we define $x_i \in \{0, 1\}$ $(\forall i \in V)$ as the decision variables, where x_i is equal to 1 iff node i is included in the solution (clique). The mathematical formulation of MRCP reads as follows :

$$\max \left\{ \frac{\sum\limits_{i=1}^{n} a_i x_i}{\sum\limits_{i=1}^{n} b_i x_i} : x_i + x_j \leq 1 : \forall (i,j) \notin E, i \neq j; \sum_{i=1}^{n} (1 - a_{ij}) x_i \geq 1 : \forall j \in V; x_i \in \{0,1\} : \forall i \in V \right\}.$$

In this model, the non-negativity of the objective function as well as the second constraints force the model to give us a maximal clique.

3 Solution Methods

In order to solve the MRCP, several methods have been proposed in the literature [4, 5]. One approach consists in reformulating the mathematical model of MRCP as a mixed integer linear programming model (MILP). For this purpose, we need to introduce additional variables y and $z_i = yx_i$ ($\forall i \in V$), where $y := 1/(\sum\limits_{i=1}^{n} b_i x_i)$. Then, one can use any standard MILP solver (such as IBM Cplex or Gurobi) to solve the problem.

Another efficient approach is based on using techniques of fractional programming. The binary search and Newton's method have already been adapted to solve MRCP [2, 3, 5]. In these methods, we start by the following model

$$\max \left\{ \sum_{i=1}^{n} a_i x_i - \lambda(\sum_{i=1}^{n} b_i x_i) : (x_1, \ldots, x_n) \in \mathcal{A} \right\}, \tag{1}$$

where $\lambda \in \mathbb{R}$ and $\mathcal{A}$ is defined as follows :

$$\left\{ (x_1, \ldots, x_n) : x_i + x_j \leq 1 : \forall (i,j) \notin E, i \neq j; \sum_{i=1}^{n} (1 - a_{ij}) x_i \geq 1 : \forall j \in V; x_i \in \{0,1\} : \forall i \in V \right\}.$$

Suppose that $P(\lambda)$ is the optimal value of (1), then both binary search and Newton's method look for λ^* such that $P(\lambda^*) = 0$. For this purpose, the binary search and Newton's method generate a sequence $\{\lambda^k\}$ that converges to λ^* (see [5] and references therein).

In this work, we propose a heuristic based on variables neighborhood search (VNS). VNS is based on using various neighborhood structures, an improvement procedure, and random perturbation (called *shaking*). This approach has been used (successfully) for solving a large variety of optimization problems [1]. In order to use this framework on MRCP, our approach starts with a greedy heuristic. The purpose is constructing maximal cliques with a good estimation of the optimal value. Then, we use several neighborhood structures that are defined by taking into account the mathematical properties and the specific structure of MRCP. The neighborhood structures consider different possibilities of moving inside solution space by either adding new nodes to or dropping some existing nodes from the current clique. Since after these changes the current solution will be no more a maximal clique, the greedy heuristic is called to repair the subgraph and reconstruct a maximal clique. A random selection is embedded inside the procedure to include the shaking mechanism. The algorithm is repeated until the stopping criterion, i.e., the maximum number of iterations, is met.

In order to evaluate the efficiency of the algorithm, we use standard instances to solve the MRCP and compare its results with the solutions of Gurobi and of the results provided by the Newton's method. According to our observations, our approach provides high-quality solutions in short computation time and has a better performance than the other methods.

Références

[1] Hansen, P. and N. Mladenovic, Variable neighbourhood search, Handbook of Metaheuristics, Dordrecht, Kluwer Academic Publishers, 2003.

[2] Ibaraki T., Parametric approaches to fractional programs, *Math Prog*, 26, 345–362, 1983.

[3] Lawler E.L., Combinatorial optimization : networks and matroids, Holt, Rinehart and Winston, New York, 1976.

[4] Moeini M., The Maximum Ratio Clique Problem : A Continuous Optimization Approach and Some New Results, *Advances in Intelligent Systems and Computing*, Vol. 359, 215–227, 2015.

[5] Sethuraman S. and S. Butenko, The Maximum Ratio Clique Problem, *Comp. Man. Sci.*, 2013.