Contributions to the robust lot-sizing problem

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Mots-clés : *robust optimization, lot-sizing, dynamic programming.*

1 Introduction

Lot-sizing optimization problems appear in a wide range of applications where products have to be made to attend demands along a planning horizon. Let us consider first the classical form of the lot-sizing problem. Let $\{1, \dots, n\}$ be a finite planning horizon. For each time period $i = 1, \dots, n$, we are given a unitary holding cost h_i, a unitary shortage cost p_i, a unitary production cost c_i, a fixed cost f_i, and a demand d_i. Then, for each time period i, $x_i \geq 0$ represents the production, $s_i \geq 0$ the stock, $r_i \geq 0$ the shortage, and y_i is a binary variable equal to 1 iff production takes place in the period. The problem is modeled as follows.

$$
\texttt{LSP} \quad
\begin{array}{lll}
\min & \displaystyle\sum_{i=1}^{n} (c_i x_i + f_i y_i + h_i s_i + p_i r_i) & \\
\text{s.t} & s_{i+1} = x_i - d_i + s_i - r_{i-1} + r_i & i = 1, \dots, n, \quad (1) \\
& x_i \leq M y_i & i = 1, \dots, n, \quad (2) \\
& y \in \{0,1\}^n, x, s, r \geq 0. &
\end{array}
$$

The cost functions considers the sum of the holding costs, shortage costs, production costs and fixed costs. Constraints (1) are equilibrium constraints linking the production, stock and shortage variables. Constraints (2) state that if we produce in a period, then we must pay a fixed cost, where M is a large predefined value.

In practical lot-sizing problems, the future demands are usually not known with precision until the current period is reached. In this work, we focus on the *robust counterpart* of `LSP`, therefore assuming that the demand uncertainty is modeled by a given polytope. This model, used in [6, 2, 4, 5], among many others, is relevant when historical data are not accurate enough to draw probabilistic distributions of the uncertain demands. While the robust *static problems*, where decisions are taken before revealing the uncertain parameters remain tractables, the situations with *adjustable problems* is far more complex. *Adjustable robust optimization* problems suppose that the uncertainty is revealed as time goes and one can adjust the values of some of the decision variables according to the current knowledge of the uncertain parameters. Hence, the adjustable optimization variables become functions of the uncertain parameters.

Robust lot-sizing problems can be classified into essentially two families of problems. In the first family, the production decisions do not depend on the uncertainty ; only the holding and shortage costs are adjustable. While $\mathcal{NP}$-hard, this family of problems has been solved exactly in some cases using decomposition techniques [6, 5], or approximately using affine decision rules [2]. The second family considers that all optimization variables are adjustable, including production variables. Besides being $\mathcal{NP}$-hard, the latter family of problems cannot be solved exactly, unless for toy problems, and the litterature only contains bounding approaches [2, 7].

This work contributes to the solution of both families of problems. In Section 2, we introduce formally the robust lot-sizing problem and the budgeted uncertainty polytope from [3]. In Section 3, we study the version with fixed production. We focus on the problem where the demand uncertainty is described by the budgeted uncertainty introduced in [3]. We propose a row-and-column generation algorithm for the problem and formulate the separation problem as dynamic program running in pseudo-polynomial time. In Section 4, we study the version with adjustable production. We propose a new lower bound inspired by the perfect information relaxation in stochastic programming. We show that the resulting optimization problem can be solved in polynomial time in several cases. For brevity, the proofs and numerical results are not reported here and we refer the interested reader to [1] and [8].

2 Robust lot-sizing

We assume that the demands are not known with precision and vary around their nominal values. Specifically, we assume here that the vector of demands d is given by $d_i(\xi) = \bar{d}_i + \hat{d}_i\xi_i$, where $\bar{d}_i$ represents the nominal demand in period i, $\hat{d}_i$ represents the maximum allowed demand deviation in period i, and ξ can take any value in the polytope defined by $\mathcal{U}^\Gamma \equiv \left\{\xi \in \mathbb{R}^n : -1 \leq \xi \leq 1,\ \sum |\xi_i| \leq \Gamma\right\}$. We introduce below the adjustable robust counterpart of LSP where the production can be adjusted to *past* demand realizations. Specifically, we let $x_i(\xi)$ be the real variable stating how much good is produced in period i for scenario ξ, while the binary variable $y_i(\xi)$ is equal to 1 iff some production occurs in period i for scenario ξ. Hence, x and y can be interpreted as functions from $\mathcal{U}^\Gamma$ to $\mathbb{R}^n_+$ and $\{0,1\}^n$, respectively. Notice that, for each time period i, x and y must depend only on the demand revealed up to time period i. This is modeled by the *non-anticipativity* constraints

$$x_i(\xi) = x_i(\xi') \text{ and } y_i(\xi) = y_i(\xi'), \qquad \forall \xi, \xi' \in \mathcal{U}^\Gamma, \mathrm{Proj}_{[1\ldots i]}(\xi) = \mathrm{Proj}_{[1\ldots i]}(\xi'),$$

where $\mathrm{Proj}_{[1\ldots i]}(\xi)$ denotes the projection of ξ on its first i components. Substituting the holding and shortage variables $s(\xi)$ and $r(\xi)$ through the robust counterpart of constraints (1), one readily verifies that the total cost of a production plan given by x and y is

$$\max_{\xi \in \mathcal{U}^\Gamma} \sum_{i=1}^{n} \left(c_i x_i(\xi) + f_i y_i(\xi) + \max\left\{ h_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j(\xi) \right), -p_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j(\xi) \right) \right\} \right) \tag{3}$$

To simplify the presentation of the algorithm presented in the next section, we split the total cost (3) into the production costs in one side, and the shortage and holding cost in the other side. Specifically, we introduce a set of adjustable variables $\varphi_i(\xi)$ which represent, for each period i and scenario ξ, the maximum between the holding cost and the shortage cost.

$\mathcal{U}^\Gamma$-LSP

$$\begin{aligned}
\min \quad & \max_{\xi \in \mathcal{U}^\Gamma} \sum_{i=1}^{n} (\varphi_i(\xi) + c_i x_i(\xi) + f_i y_i(\xi)) \\
\text{s.t.} \quad & \varphi_i(\xi) \geq h_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j \right), \quad \xi \in \mathcal{U}^\Gamma, i = 1, \ldots, n, && (4) \\
& \varphi_i(\xi) \geq -p_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j \right), \quad \xi \in \mathcal{U}^\Gamma, i = 1, \ldots, n && (5) \\
& x_i(\xi) \leq M y_i(\xi), \quad i = 1, \ldots, n, && (6) \\
& \begin{matrix} x_i(\xi) = x_i(\xi') \\ y_i(\xi) = y_i(\xi') \end{matrix}, \quad \xi, \xi' \in \mathcal{U}^\Gamma, \mathrm{Proj}_{[1\ldots i]}(\xi) = \mathrm{Proj}_{[1\ldots i]}(\xi'), i = 1, \ldots, n, && (7) \\
& y(\xi) \in \{0,1\}^n, x(\xi), \varphi(\xi) \geq 0, \quad \xi \in \mathcal{U}^\Gamma.
\end{aligned}$$

Constraints (4) and (5) impose that φ models the aforementioned costs. Constraints (7) are the non-anticipativity constraints mentioned previously. The other constraints are similar to those of problem LSP.

3 Fixed production

We assume in this section that x and y do not depend on ξ, so that the non-anticipativity constraints (7) can be relaxed. Introducing an auxiliary variable θ to represent $\max\limits_{\xi\in\mathcal{U}^\Gamma} \sum\limits_{i=1}^{n} \varphi_i(\xi)$, we obtain the following MILP denoted $\mathcal{U}^\Gamma$-`LSP-fix`.

$$
\mathcal{U}^\Gamma\text{-}\texttt{LSP-fix} \qquad
\begin{aligned}
\min \quad & \theta + \sum_{i=1}^{n}(c_i x_i + f_i y_i) \\
\text{s.t.} \quad & \theta \ge \sum_{i=1}^{n} \varphi_i(\xi), \quad \xi \in \mathcal{U}^\Gamma, && (8)\\
& \varphi_i(\xi) \ge h_i\left(\sum_{j=1}^{i}\xi_j - \sum_{j=1}^{i} x_j\right), \quad \xi \in \mathcal{U}^\Gamma, i = 1,\dots,n, && (9)\\
& \varphi_i(\xi) \ge -p_i\left(\sum_{j=1}^{i}\xi_j - \sum_{j=1}^{i} x_j\right), \quad \xi \in \mathcal{U}^\Gamma, i = 1,\dots,n && (10)\\
& x_i \le M y_i, \quad i = 1,\dots,n, \\
& y \in \{0,1\}^n, x, \varphi \ge 0
\end{aligned}
$$

Problem $\mathcal{U}^\Gamma$-`LSP-fix` contains infinite numbers of variables and constraints. Hence, we tackle the problem through the following row-and-column generation algorithm. Let $\mathcal{U}^*$ be a finite subset of $\mathcal{U}^\Gamma$. Given a feasible solution (x^*, θ^*) to $\mathcal{U}^*$-`LSP-fix`, one checks the feasibility of (x^*, θ^*) for $\mathcal{U}^\Gamma$-`LSP-fix` by solving the adversarial problem

$$
\max_{\xi\in\mathcal{U}^\Gamma} \sum_{i=1}^{n} \max\left\{ h_i\left(\sum_{j=1}^{i}\xi_j - \sum_{j=1}^{i} x_j^*\right), -p_i\left(\sum_{j=1}^{i}\xi_j - \sum_{j=1}^{i} x_j^*\right)\right\}; \qquad (11)
$$

we denote the objective function of (11) as $g(\xi)$ for short. Let ξ^* be the optimal solution for the adversarial problem. If $g(\xi^*) > \theta^*$, then $\mathcal{U}^* \leftarrow \mathcal{U}^* \cup \{\xi^*\}$, and the corresponding optimization vector $\varphi(\xi^*)$ and constraints (8)–(10) are added to $\mathcal{U}^*$-`LSP-fix`. We can prove the following.

Theorem 1. *Problem (11) can be solved by a dynamic programming algorithm in* $\mathcal{O}(n^2\Gamma \max\limits_{1\le i\le n} \widehat{d_i})$.

Theorem 2. *Suppose that* $\min\{h_n, p_n\} > 0$ *and that*

$$
\frac{2\max\limits_{i=1,\dots,n}\{h_i, p_i\}}{\min\{h_n, p_n\}} \le \mathcal{P}(n)
$$

where $\mathcal{P}(n)$ *is a polynomial in* n. *There exists a FPTAS for (11). Moreover, if* $f = 0$ *then there also exists a FPTAS for* $\mathcal{U}^\Gamma$*-LSP.*

In addition to these theoretical results, our numerical results (reported in [1]) show that the row-and-column generation algorithm performs very well on instances from the literature.

4 Adjustable production

We consider in this section the problem where x is a vector of adjustable variables and that $\mathcal{U} \subset \mathbb{R}^n_+$ is an arbitrary polytope defined by m inequalities. Since the non-anticipativity constraints (7) make the problem untractable, they have been removed from $\mathcal{U}$-LSP, yielding the *perfect information* relaxation.

$$
\begin{aligned}
& & \min \quad & \theta + \max_{\xi \in \mathcal{U}} \sum_{i=1}^{n} (c_i x_i(\xi) + f_i y_i(\xi)) & \\
& \mathcal{U}\text{-LSP-pi} & \text{s.t.} \quad & \theta \geq \sum_{i=1}^{n} \varphi_i(\xi), & \xi \in \mathcal{U}, \\
& & & \varphi_i(\xi) \geq h_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j \right), & \xi \in \mathcal{U}, i = 1, \dots, n, \\
& & & \varphi_i(\xi) \geq -p_i \left(\sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i} x_j \right), & \xi \in \mathcal{U}, i = 1, \dots, n \\
& & & x_i(\xi) \leq M y_i(\xi), & i = 1, \dots, n, \\
& & & y(\xi) \in \{0,1\}^n, x(\xi), \varphi(\xi) \geq 0, & \xi \in \mathcal{U}.
\end{aligned}
$$

We consider three variants of the problem, depending on whether y is adjustable and f positive. First, we consider the general problem and prove that it can be solved in polynomial time.

Theorem 3. *Problem $\mathcal{U}$-LSP-pi amounts to solve a linear program with $m + n^3$ constraints and $2n$ variables.*

If the problem does not contain fixed costs, it can be solved as a smaller linear program.

Theorem 4. *If $f = 0$, then problem $\mathcal{U}$-LSP-pi amounts to solve a linear program with m constraints and n variables.*

The last case we consider is obtained from $\mathcal{U}$-LSP-pi by enforcing that y does not depend on ξ, which is obtained by adding the constraint $y(\xi) = y(\xi')$ for each $\xi, \xi' \in \mathcal{U}$.

Theorem 5. *If y does not depend on ξ, problem $\mathcal{U}$-LSP-pi is $\mathcal{NP}$-hard in general. Nevertheless, if $\mathcal{U} = \mathcal{U}^\Gamma$, the problem can be solved in polynomial time, by solving n^2 shortest path problems on a directed and acyclic graph with n nodes.*

In addition to these theoretical results, our numerical results (reported in [8]) show that the resulting problems are solved orders of magnitude faster than the dual bounds proposed in [7], while obtaining better bounds in many cases.

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