On the multi-terminal vertex separator problem

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Keywords : *Vertex separator problem, Combinatorial optimization, Polytope composition, Column generation, Branch-and-Cut, Branch-and-Price.*

1 Introduction

Let $G = (V \cup T, E)$ be a simple graph with $V \cup T$ the set of vertices, where T is a set of k distinguished vertices called *terminals*, and E the set of edges. A multi-terminal vertex separator in G is a subset $S \subseteq V$ such that each path between two terminals intersects S. Given a weight function $w : V \to \mathbb{N}$, the multi-terminal vertex separator problem (MTVSP) consists in finding a multi-terminal vertex separator of minimum weight. The MTVSP can be solved in polynomial time when $|T| = 2$ and it is NP-hard when $|T| \geq 3$ [1] [3]. The MTVS has applications in different areas like VLSI design, linear algebra, connectivity problems and parallel algorithms. In this paper we consider the MTVSP from a polyhedral point of view, we describe several facet defining inequalities and propose a Branch-and-Cut algorithm for the problem. Then, we study a composition (decomposition) technique for the multi-terminal vertex separator polytope in graphs that are decomposable by one-node cutsets. If G decomposes into G_1 and G_2, we show that the multi-terminal vertex polytope of G can be described from two linear systems related to G_1 and G_2. As consequence, we obtain a procedure to construct this polytope in graphs that are recursively decomposed. Finally, we propose an extended formulation and derive a Branch-and-Price algorithm.

A terminal path in $G = (V \cup T, E)$ is a path between two terminals. Let Γ be the set of all the terminal paths in G. Let $x \in \{0,1\}^V$ such that for all $v \in V$, x_v is equal to 1 if vertex v belongs to the separator, 0 otherwise. The MTVSP is equivalent to the following ILP

$$\min \sum_{v \in V} x_v$$

$$\sum_{v \in P_{tt'}} x_v \geq 1 \qquad \forall P_{tt'} \in \Gamma, \tag{1}$$

$$x_v \in \{0,1\} \qquad \forall v \in V. \tag{2}$$

In what follows we discuss a polyhedral approach for the MTVSP.

2 Polyhedral approach

Let $P(G,T)$ be the convex hull of the solutions satisfying (1)-(2). We have that $P(G,T)$ is full dimensional and for all $v \in V$, inequality $x_v \leq 1$ defines a facet of $P(G,T)$. Moreover, we have the following results

Theorem 1

1. *For $v \in V$, inequality $x_v \geq 0$ defines facet of $P(G,T)$ if and only if, v does not belong to a terminal path $P_{tt'}$ containing exactly two internal vertices.*

2. *Inequality (1) associated with a terminal path $P_{tt'}$ defines a facet of $P(G,T)$ if and only if $P_{tt'}$ is minimal and no vertex $v \in V \setminus P_{tt'}$ is adjacent to a terminal $t \in T \setminus \{t,t'\}$ and to two vertices of $P_{tt'}$.*

Now we give further valid inequalities that may define facets for $P(G,T)$.

A *star tree* is a tree where the pending nodes are the terminal nodes of the tree, and all the other (nonterminal) nodes, different from the root node v_r are of degree two.

Theorem 2 *If $H_q = (V_{H_q} \cup T_{H_q}, E_{H_q})$ is a star tree of G with root v_r, then inequality*

$$x(V_{H_q} \setminus \{v_r\}) + (q-1)x_{v_r} \geq q-1 \tag{3}$$

is valid for $P(G,T)$. Moreover it defines a facet for $P(G,T)$ under some conditions.

A *terminal tree* $R_q = (V_{R_q} \cup T_{R_q}, E_{R_q})$, where T_{R_q} is a set of q terminals, is a tree, where T_{R_q} is the set of leaves. Let $d_{R_q}(v)$ be the degree of v in R_q.

Theorem 3 *Given a terminal tree R_q, the following inequality is valid for $P(G,T)$*

$$\sum_{v \in R_q} (d_{R_q}(v)-1)x_v \geq q-1. \tag{4}$$

A *terminal cycle* $J_q = (C \cup T_{J_q}, E_{J_q})$, where T_{J_q} is a set of q terminals $\{t_1, \ldots, t_q\}$, is a graph given by a cycle $C = \{v_1, \ldots, v_q\}$ of q vertices and q disjoint edges, $e_{t_1}, \ldots, e_{t_q}$ between the vertices of C and the terminals of T_{J_q}.

Theorem 4 *If $J_q = (C \cup T_{J_q}, E_{J_q})$ is a terminal cycle of G, then the following inequality*

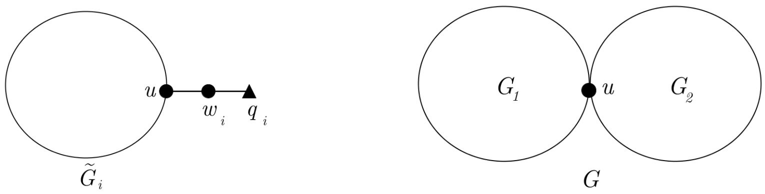
$$x(C) \geq \lceil \frac{q}{2} \rceil \tag{5}$$

is valid for $P(G,T)$. Moreover, it defines a facet for $P(G,T)$ under some conditions.

Theorem 5 *Inequalities* (1) *and star tree inequalities can be separated in polynomial time.*

3 Composition of polyhedra

We study a composition (decomposition) technique for the multi-terminal vertex separator polytope in graphs that are decomposable by one-node cutsets. Given a graph $G = (V \cup T, E)$ and two subgraphs $G_1 = (V_1 \cup T_1, E_1)$ and $G_2 = (V_2 \cup T_2, E_2)$, graph G is called a $1-sum$ of G_1 and G_2 if $V = V_1 \cup V_2$, $T = T_1 \cup T_2$, $|T_1 \cap T_2| = 0$, $V_1 \cap V_2 = \{u\}$. Let $\widetilde{G}_i = (\widetilde{V}_i \cup \widetilde{T}_i, \widetilde{E}_i)$ be the graph obtained from G_i, for $i = 1,2$, by adding a node w_i, a terminal q_i and the edges q_iw_i, w_iu.



Theorem 6 *The linear inequalities describing $P(\widetilde{G}_i, \widetilde{T}_i)$ can be partitioned as follows*

$$\sum_{v \in V_i \setminus \{u\}} a^i_j(v)x(v) \geq \alpha^i_j \quad \forall j \in I^i \tag{6}$$

$$\sum_{v \in V_i \setminus \{u\}} a'^i_j(v)x(v) + x(u) \geq \alpha'^i_j \quad \forall j \in I'^i \tag{7}$$

$$\sum_{v \in V_i} b^i_j(v)x(v) + x(w_i) \geq \beta^i_j \quad \forall j \in J^i \tag{8}$$

$$x(v) \leq 1 \quad \forall v \in \widetilde{V}^i \tag{9}$$

$$x(v) \geq 0 \quad \forall v \in \widetilde{V}^i \tag{10}$$

Theorem 7 *The linear inequalities describing $P(G,T)$ are as follow*

$$\sum_{v \in V_i \setminus \{u\}} a_j^1(v)x(v) \geq \alpha_j^1 \qquad \forall j \in I^1 \tag{11}$$

$$\sum_{v \in V_1 \setminus \{u\}} a_j'^1(v)x(v) + x(u) \geq \alpha_j'^1 \qquad \forall j \in I'^1 \tag{12}$$

$$\sum_{v \in V_i \setminus \{u\}} a_j^2(v)x(v) \geq \alpha_j^2 \qquad \forall j \in I^2 \tag{13}$$

$$\sum_{v \in V_2 \setminus \{u\}} a_j'^2(v)x(v) + x(u) \geq \alpha_j'^2 \qquad \forall j \in I'^2 \tag{14}$$

$$\sum_{p=1}^{2} \sum_{v \in V_i} b_{j_p}^i(v)x(v) - x(u) \geq \sum_{p=1}^{2} \beta_{j_p}^i - 1 \qquad \forall j_1 \in J^1, \forall j_2 \in J^2 \tag{15}$$

$$x(v) \leq 1 \qquad \forall v \in V \tag{16}$$

$$x(v) \geq 0 \qquad \forall v \in V \tag{17}$$

Inequalities (11) and (13) represent inequalities (6) and inequalities (12) and (14) represent inequalities (7). Inequalities (15) are called the mixed inequalities.

A linear system $Ax \leq b$ is *total dual integral* if for all $c \in \mathbb{Z}^n$, the problem $\{ \max \ c^\top x \ : \ Ax \leq b \}$ has a feasible solution and there is an integer optimal dual solution.

Theorem 8 *For any star tree, the linear system given by* (1), (3) *and trivial inequalities is total dual integral.*

Corollary 3.1 *From Theorems 7 and 8, for any terminal tree, the polytope given by* (1), (4) *and trivial inequalities is integral.*

4 Extended formulation

In this section, we introduce an extended formulation for the MTVSP and develop a Branch-and-Price algorithm to solve it. For a terminal $t \in T$, a *isolating-separator* $S^t \subseteq V$ of G is a set of vertices that intersects all paths between t and terminals of $T \setminus \{t\}$. For a terminal $t \in T$, let $\mathcal{S}^t$ be the set of all isolating-separators in G associated with t. Let $\mathcal{S}$ be the set of all isolating-separators in G. Let $x \in \{0,1\}^{\mathcal{S}}$ and $y \in \{0,1\}^V$ such that

$$x^S = \begin{cases} 1 & \text{if isolating-separator } S \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } S \in \mathcal{S}$$

$$y_v = \begin{cases} 1 & \text{if } v \in V \text{ belongs to the separator,} \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V$$

Each $S \in \mathcal{S}$ is defined by the vectors $a^S \in \{0,1\}^{V \cup T}$, $\overline{a}^S \in \{0,1\}^V$ and $b^S \in \{0,1\}^T$, as follows

$$a_{v,t}^S = \begin{cases} 1 & \text{if } v \text{ belongs to } S \text{ and } S \in \mathcal{S}^t, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V, \ t \in T$$

$$\overline{a}_v^S = \begin{cases} 1 & \text{if } v \text{ belongs to } S, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V$$

$$b_t^S = \begin{cases} 1 & \text{if } S \in \mathcal{S}^t, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } t \in T$$

The MTVSP is equivalent to the following integer linear formulation

$$\min \sum_{v\in V} y_v \tag{18}$$

$$y_v - \sum_{S\in\mathcal{S}} a_{v,t}^S x^S \geq 0 \qquad \forall t \in T, \quad \forall v \in V, \tag{19}$$

$$-y_v + \sum_{S\in\mathcal{S}} \overline{a}_v^S x^S \geq 0 \qquad \forall v \in V, \tag{20}$$

$$\sum_{S\in\mathcal{S}^t} b_t^S x^S = 1 \qquad \forall t \in T, \tag{21}$$

$$x^S \geq 0 \qquad \forall\ S \in \mathcal{S}, \tag{22}$$

$$y_v \in \{0,1\} \qquad \forall\ v \in V. \tag{23}$$

The pricing problem aims at generating an isolating-separator S^{t^*} associated with the terminal $t^* \in T$. In the following Table, D and R represent the Dimacs and random instances, respectively. The Columns, Cols and No represent, the number of variables generated and the number of nodes in the branching tree, respectively.

Instance				Branch-and-Price algorithm				Branch-and-Cut algorithm						
	n	m	$\|T\|$	Cols	No	Gap	CPU	(1)	(3)	(4)	(5)	No	Gap	CPU
D	74	624	6	308	34	0.24	**0.22**	64	51	5	1	23	19.20	3.39
D	87	835	6	419	31	0.40	**0.31**	31	147	26	3	26	28.50	8.58
D	95	778	6	475	25	0.57	**0.24**	35	38	26	15	1	0.00	1.26
D	100	2967	8	871	33	0.63	**1.00**	56	31	4	6	1	0.00	1.55
D	128	804	8	4791	63	0.39	23.24	103	127	4	2	29	28.10	**5.75**
D	128	10426	8	863	35	0.60	3.38	31	22	11	8	1	0.00	**1.31**
D	144	5224	8	3995	35	0.56	25.41	85	20	9	9	1	0.00	**1.44**
D	188	3920	8	904	41	0.60	**1.53**	116	63	2	13	1	0.00	2.69
D	196	8399	8	1801	37	0.70	5.39	56	19	12	11	1	0.00	**1.24**
D	197	3952	8	606	27	0.64	0.72	71	11	10	3	1	0.00	**0.58**
D	256	12674	8	4888	43	0.65	**43.69**	88	27	5	11	1	0.00	2.10
R	50	513	7	552	27	0.64	**0.46**	48	33	31	6	1	0.00	1.24
R	70	993	7	678	29	0.65	0.80	44	18	19	4	1	0.00	**0.64**
R	100	1986	7	561	27	0.64	**0.61**	45	13	12	5	1	0.00	1.02
R	300	17793	7	816	29	0.65	4.42	67	8	8	2	1	0.00	**2.31**
R	600	70674	7	6624	29	0.65	285.23	23	10	9	5	1	0.00	**4.04**
R	700	96436	7	1381	27	0.64	48.41	48	7	5	1	1	0.00	**3.96**

TAB. 1: Results associated with the Branch-and-Cut and the Branch-and-Price algorithms.

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