The multi-terminal vertex separator problem

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1 Introduction

Let $G = (V \cup T, E)$ be a simple graph with $V \cup T$ the set of vertices, where T is a set of k distinguished vertices called *terminals*, and E the set of edges. The multi-terminal vertex separator is a subset $S \subseteq V$ such that each path between two terminals intersects S. Given a weight function $w : V \to \mathbb{N}$, the multi-terminal vertex separator problem (MTVSP) consists in finding a multi-terminal vertex separator of minimum weight. The problem can be solved in polynomial time when $|T| = 2$ and it is NP-hard when $|T| \geq 3$ [1] [3]. The MTVS problem has applications in different areas like VLSI design, linear algebra, connectivity problems and parallel algorithms. The MTVSP is equivalent to the following integer linear program

$$\min \sum_{v \in V} x_v$$

$$\sum_{v \in P_{tt'}} x_v \geq 1 \qquad \forall \text{ path } P_{tt'} \text{ between two terminals } t, t' \in T, \tag{1}$$

$$x_v \in \{0,1\} \qquad \forall v \in V. \tag{2}$$

Let $P(G,T)$ be the convex hull of the solutions of the above formulation, that is,

$$P(G,T) = conv(x \in \{0,1\}^V \mid \ x \text{ satisfies } (1))$$

In this work, we first show that the problem is NP-hard, we then discuss the polyhedral structure of $P(G,T)$. We describe some valid inequalities and characterize when these inequalities define facets. Then, we study a composition (decomposition) technique for the multi-terminal vertex separator polytope in graphs that are decomposable by one-node cutsets. If G decomposes into G_1 and G_2, we show that the multi-terminal vertex polytope of G can be described from two linear systems related to G_1 and G_2. As consequence, we obtain a procedure to construct new valid inequalities for our polytope. Using these results we develop a Branch-and-Cut algorithm for the problem. Moreover, we introduce a new extended formulation for the MTVSP and develop a Branch-and-Price algorithm to solve it, as follows. For a terminal $t \in T$, an *isolating-separator* $S^t \subseteq V$ of G is a set of vertices that intersects all paths between t and terminals of $T \setminus \{t\}$. For a terminal $t \in T$, let $\mathcal{S}^t$ be the set of all isolating-separators in G associated with t and $\mathcal{S} = \bigcup_{t \in T} \mathcal{S}^t$. Let $x \in \{0,1\}^{\mathcal{S}}$ and $y \in \{0,1\}^V$ defined as follows

$$x^S = \begin{cases} 1 & \text{if isolating-separator } S \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } S \in \mathcal{S}$$

$$y_v = \begin{cases} 1 & \text{if } v \in V \text{ belongs to the separator,} \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V$$

Each separator $S \in \mathcal{S}$ is defined by the vectors $a^S \in \{0,1\}^{V \cup T}$, $\overline{a}^S \in \{0,1\}^V$ and $b^S \in \{0,1\}^T$, as follows

$$a^S_{v,t} = \begin{cases} 1 & \text{if } v \text{ belongs to } S \text{ and } S \in \mathcal{S}^t, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V,\ t \in T$$

$$\overline{a}^S_v = \begin{cases} 1 & \text{if } v \text{ belongs to } S, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } v \in V$$

$$b^S_t = \begin{cases} 1 & \text{if } S \in \mathcal{S}^t, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for all } t \in T$$

The MTVSP is equivalent to the following integer linear formulation

$$\min \quad \sum_{v \in V} y_v \tag{3}$$

$$y_v - \sum_{S \in \mathcal{S}} a^S_{v,t} x^S \geq 0 \qquad \forall t \in T, \quad \forall v \in V, \qquad (u^t_v) \tag{4}$$

$$-y_v + \sum_{S \in \mathcal{S}} \overline{a}^S_v x^S \geq 0 \qquad \forall v \in V, \qquad (\lambda_v) \tag{5}$$

$$\sum_{S \in \mathcal{S}^t} b^S_t x^S = 1 \qquad \forall t \in T, \qquad (\eta_t) \tag{6}$$

$$x^S \geq 0 \qquad \forall\ S \in \mathcal{S}, \tag{7}$$

$$y_v \in \{0,1\} \qquad \forall\ v \in V. \tag{8}$$

The pricing problem aims at generating an isolating-separator S^{t^*} associated with $t^* \in T$.

Instances	n	m	$\lvert T \rvert$	Cols	Nodes	Gap %	CPU(s)
Dimacs	206	3576	8	961	29	0.56	1.39
Dimacs	211	3573	8	927	31	0.67	1.10
Dimacs	211	4132	8	765	27	0.64	1.36
Dimacs	256	12674	8	4888	43	0.65	43.69
Random	50	511	7	552	27	0.64	0.46
Random	70	993	7	678	29	0.65	0.80
Random	100	1985	7	561	27	0.64	0.61
Random	300	17792	7	816	29	0.65	4.42
Random	600	70673	7	6624	29	0.65	285.23
Random	700	96432	7	1381	27	0.64	48.41

TAB. 1: Results associated with the Branch-and-Price algorithm.

References

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