The k-matching cover problem: min-max relation and algorithm

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1 Introduction

Let $G(V,E)$ be a graph. A *matching* M of G is a set of edges where no two edges share a same node. A matching M is *maximum* if it is of maximum size over all matchings in G and it is *perfect* if each node $u \in V$ is incident to an edge in M . A *k-matching cover* of G is a set of k disjoint matchings $M_1,\dots,M_k$ of G where each node of V is incident to at least one edge from $\cup_{i=1}^{k} M_i$. A *minimum* matching cover is a k-matching cover with a minimum number k of matchings. The *minimum matching cover problem* is the problem of finding such a minimum matching cover. Obviously the problem reduces to the question of whether a graph $G(V,E)$ has a k-matching cover of size k for all $k = 1,...,m$. When $k = 1$, this is equivalent to a perfect matching problem. When $k \geq 2$, the problem is known to be polynomial [5] because it can be modelled as a union of k matching matroids and it can thus be solved through matroid intersection (see for instance [4]). In [5], the authors proposed an alternative faster approach based on Edmonds-Gallai Theorem and technical augmentation algorithms. Unfortunately, their algorithm contains flaws. We show in this paper that actually Edmonds-Gallai Theorem can be used in conjonction with simple flow arguments to give a nice min-max relation for the problem and a new algorithm : the min-max relation can be seen as an extension of Hall's Theorem when $k \geq 2$.

Theorem 1. *(Hall's theorem 1937) Let $G(X,Y,E)$ be a bipartite graph. Then G has a perfect matching if and only if $|S| \leq |N(S)|$, for each $S \subseteq X$ or $S \subseteq Y$.*

2 Results

Indeed, for bipartite graphs, we can extend Hall's Theorem to k-matching cover as follows :

Theorem 2. *Let $G(X,Y,E)$ be a bipartite graph and let $k \in \mathbb{N} \setminus \{0\}$. There exists a k-matching cover of G if and only if $\forall S \subseteq X$ and $\forall S \subseteq Y$, $|S| \leq |N(S)| \cdot k$.*

Quite surprisingly, this result hold for general graphs if we slightly adapt the theorem as long as $k \geq 2$, as shown by the following theorem. We prove this more general version. The previous theorem then follows as a simple corollary.

Theorem 3. *Let $G(V,E)$ be a graph with no perfect matching. Let k be an integer such that $k \geq 2$. There exists a k-matching cover of G if and only if for all stable set S of G, $|S| \leq |N(S)| \cdot k$.*

The proof of this theorem relies on Edmonds-Gallai structure theorem (see for instance [3]), the max-flow min-cut theorem and simple combinatorial arguments. Let us recap the flavour of

Edmonds-Gallai Theorem : let $G(V,E)$ be a graph and let A,C,D be the subsets of V defined as follow: $D = \{v \in V|$ there exists a maximum matching M of G missing $v\}$, $A = \{v \in V | v$ has a neighbor in D and $v \notin D\}$, and $C = V \setminus (D \cup A)$.

Edmonds-Gallai [1, 2] proved that we can find these sets in polynomial time via Edmonds' matching algorithm. We exploit some of the properties of these sets (see an example in Fig. 1) to prove that the three following statements (a), (b), (c) are equivalent (we denote by D_1 the set of odd connected components of $G[D]$ of size 1).

(a) $\exists$ k-matching cover of $V(G)$.

(b) $\forall S$ stable set of G, then $|S| \leq |N(S)| \cdot k$.

(c) $\exists$ a flow of size $|D_1|$ in the network $(G'(V',E'),c)$ with $V' = \{s,t\} \cup D_1 \cup A$, $E' := \{e = (u,v) \in E : \forall u \in D_1, v \in A\} \cup \{(s,u) : \forall u \in D_1\} \cup \{(v,t) : \forall v \in A\}$, and $c(e) = 1$ if $e = (s,u)$ for some u, $c(e) = k$ if $e = (v,t)$ for some v and $c(e) = +\infty$ otherwise.

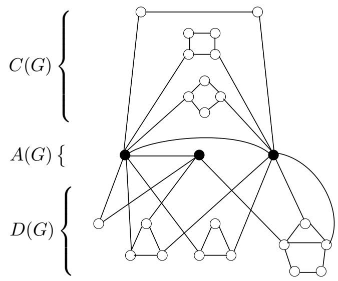

FIG. 1: The Edmonds-Gallai decomposition of a graph G.

3 Conclusion

In this paper, we give an extension of Hall's Theorem for k-matching cover in general graphs (without perfect matching) when $k \geq 2$. It can be turned into an efficient polytime algorithm combining Edmonds'algorithm for matching and any efficient algorithm for network flows.

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